TIME-DEPENDENT GENERALIZED PERTURBATION METHODS FOR BURN-UP ANALYSIS

by

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FORWORD

Burn-up analysis plays a fundamental role in the assessment of the parameters which enter directly into the burn-up and decay equations, the knowledge of the correct values of which is directly connected with the ability of predicting the reactor life behaviour. In order then of performing an optimal exploitation of the information contained in the experimental quantities measured, i.e. the amounts of the various burnt-up and built-up fissile, fertile and fission product isotopes, the knowledge of the sensitivities of these integral quantities with respect to such parameters (i.e., capture and fission cross-sections and decay constants) should be known. In simple cases (of very few variables) for which an analytical solution may be easily determined, the evaluation of the sensitivities, corresponding to the first derivative of a given isotope density at a given final time with respect to the various parameters, is straight-forward. In those cases, however, in which the number of variables and the complexity of the problem renders impractical the analytical evaluation of the derivatives, a method could be used, as shown in the present paper, exploiting the time-dependent generalized perturbation techniques [1]. What is here required are the real and adjoint solutions of the burn-up equations at umperturbed conditions, which may be obtained numerically once for all the sensitivity coefficients requested for a given integral quantity. In the illustrative example at the end, the correspondance is shown of the proposed perturbative method with

the evaluation of the first derivative of the integral quantity considered with respect to the various parameters.

THEORY

Having defined the burn-up and decay matrix

$$\mathcal{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1I} \\ a_{21} & a_{22} & \cdots & a_{2I} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ a_{I1} & a_{I2} & \cdots & a_{II} \end{vmatrix} \qquad (1)$$

relevant to I isotopes with density $n_1, n_2, \ldots, n_I$ and where a_{ij} represent given coefficients (°), the burn-up equation may be written:

$$\frac{d\underline{n}}{dt} = \mathcal{A}\underline{n}(t). \qquad (2)$$

If the initial density $\underline{n}_o$ at time t_o is given and the matrix $\mathcal{A}$ results time-independent, we can write the formal solution:

$$\underline{n}(t) = e^{\mathcal{A}(t-t_o)}\underline{n}_o. \qquad (3)$$

(°) In the a_{ij} coefficients the following terms may generally be present:

a) λ_i (decay constant of isotope i);

b) $\Phi\sigma_{a,i}$ (flux by absorption cross-section of isotope i);

c) $\Phi\sigma_{f,i}$ (flux by fission cross-section of isotope i);

d) $K_{ij}\Phi\sigma_{f,i}$ (K_{ij} = fraction per fission of isotope i of the fission product j).

Let us now consider the integral quantity

$$Q=\underline{h}^T\,\underline{n}(t_F) \qquad (t_F > t_o) \tag{4}$$

where

$$\underline{h}^T=|h_1\ h_2\ \ldots\ h_I|\ , \tag{5}$$

h_1 representing assigned values (generally = 0 or 1), and let us write with respect to Eqn. (2) the adjoint equation, by which the importance function results defined. We obtain:

$$-\frac{d\underline{n}^*(t)}{dt} = \mathcal{A}^T\underline{n}^*(t), \tag{6}$$

where $\mathcal{A}^T$ is the transposed matrix of $\mathcal{A}$, with the (final) condition at time t_F:

$$\underline{n}^*(t_F) = \underline{h}\ . \tag{7}$$

The solution will be:

$$\underline{n}^*(t)=\underline{h}^T e^{-\mathcal{A}^T(t-t_F)}. \tag{8}$$

If after the time t_o we now introduce a perturbation $\delta\mathcal{A}$ into the matrix $\mathcal{A}$, a change δQ of the quantity Q will generally follow. By making use of the expression (11) of Ref. [1], which may be directly extended from the neutron to the nuclide field, and recalling that in our case $\delta\mathcal{A}=0$ for $t < t_o$, we can write the exact perturbation expression:

$$\delta Q = \int_{t_o}^{t_F} dt\ \underline{n}^{*T}(t)\,\delta\mathcal{A}\,\underline{n}'(t) \tag{9}$$

where $\underline{n}'$ represents the isotopic density after the perturbation.

Let us now suppose that at times t_o, t_1, ..., t_F the densities of the various isotopes are altered by an external operation (material removal, addition, substitution, etc.) and that a change is

induced also into matrix $\mathcal{A}$ (which, however, at different times is supposed time-independent). In this case, for the generic time-interval $(t_{\ell-1}, t_\ell)$, we can write the solution:

$$\underline{n}^{\ell}(t) = e^{\mathcal{A}_{\ell-1}(t-t_{\ell-1})}(\mathcal{U}-\mathcal{M}_{\ell-1})\, e^{\mathcal{A}_{\ell-2}(t_{\ell-1}-t_{\ell-2})}(\mathcal{U}-\mathcal{M}_{\ell-2}) \ldots$$

$$\ldots\ldots e^{\mathcal{A}_1(t_1-t_o)}(\mathcal{U}-\mathcal{M}_o)\,\underline{n}_o \qquad (10)$$

where

$$(\mathcal{U}-\mathcal{M}_\ell) = \begin{vmatrix} (1-m_1^\ell) & & & 0 \\ & (1-m_2^\ell) & \cdots & \\ & & & \\ 0 & & & (1-m_I^\ell) \end{vmatrix} \qquad (11)$$

is a diagonal matrix in which the coefficients m_i^ℓ represent the fractional variations of the generic i-th isotope at time t_ℓ . At this point the problem remains of determining the importance functions corresponding to the density $\underline{n}^{\ell}(t)$ as given by Equ. (10). For this purpose the following argument may be followed: the importance $n_i^*(t)$ of a nuclide i is defined as the contribution given by such nuclide, introduced into the system at time t, to the "measurement" Q at the final time t_F and since at every intermediate time t_ℓ we assume a reduction (or increase) by a fraction m_i^ℓ of such nuclide, its importance at the time t_ℓ^- (i.e., immediately preceding t_ℓ) undergoes a corresponding decrease (or increase) with respect to time t_ℓ^+, i.e. immediately following t_ℓ) by the same fraction m_i^ℓ. Therefore the solution adjoint to the expression given by Equ. (10) in the interval $(t_{\ell-1}, t_\ell)$ results:

$$\underline{n}^{*\ell T} = \underline{h}^T(\mathcal{U}-\mathcal{M}_L)\, e^{-\mathcal{A}_L^T(t_{L-1}-t_L)}(\mathcal{U}-\mathcal{M}_{L-1})\, e^{-\mathcal{A}_{L-1}^T(t_{L-2}-t_{L-1})} \ldots$$

$$\ldots (\mathcal{U}-\mathcal{M}_\ell)\, e^{-\mathcal{A}_\ell^T(t-t_\ell)} \qquad (12)$$

and the perturbation δQ will be given by Equ. (9) which, when linearized by setting the unperturbed density $\underline{n}$ in place of $\underline{n}'$, may be written:

$$\delta Q = \sum_{\ell} \int_{t_{\ell-1}}^{t_\ell} \underline{n}^{*\ell T}(t)\, \delta \mathcal{A}_\ell\, \underline{n}^{\ell}(t)\, dt \quad . \tag{13}$$

If we now suppose that:

$$\mathcal{A}_\ell = \mathcal{A}$$
$$\mathcal{M}_\ell = \mathcal{M}$$
$$t_\ell - t_{\ell-1} = \Delta$$

and that at times t_o and t_F there are no external alterations, Equs. (10) and (12) may be written:

$$\begin{cases} \underline{n}^{\ell}(t) = e^{\mathcal{A}(t-t_{\ell-1})} \left[(\mathcal{U} - \mathcal{M})\, e^{\mathcal{A}\Delta} \right]^{\ell-1} \underline{n}_o \\ \underline{n}^{*\ell T}(t) = \underline{h}^T \left[e^{-\mathcal{A}^T \Delta} (\mathcal{U} - \mathcal{M}) \right]^{\ell-1} e^{-\mathcal{A}^T (t-t_i)} \end{cases} \quad . \tag{14}$$

EXAMPLE

Let us consider the two equation system:

$$\begin{cases} \dfrac{dn_1}{dt} = a_{11} n_1 \\ \dfrac{dn_2}{dt} = a_{21} n_1 + a_{22} n_2 \end{cases} \qquad \text{(E.1)}$$

If we indicate by n_{01} and n_{02} the initial conditions at time t_o, the solutions result (for semplicity, supposing in this case that $a_{11} \neq a_{22}$):

$$\begin{cases} n_1(t) = n_{01}\, e^{a_{11}(t-t_o)} \\ n_2(t) = \dfrac{a_{21}\, n_{01}}{a_{11} - a_{22}}\, e^{a_{11}(t-t_o)} + \left(n_{02} - \dfrac{a_{21}\, n_{01}}{a_{11} - a_{22}}\right) e^{a_{22}(t-t_o)} . \end{cases} \qquad \text{(E.2)}$$

The system adjoint to (E.1) is given by the equations

$$\begin{cases} -\dfrac{dn_1^*}{dt} = a_{11}\, n_1^* + a_{21}\, n_2^* \\ -\dfrac{dn_2^*}{dt} = a_{22}\, n_2^* , \end{cases} \qquad \text{(E.3)}$$

the solutions of which are:

$$\begin{cases} n_1^*(t) = -\dfrac{a_{21} n_{o2}^*}{a_{11}-a_{22}} e^{-a_{22}(t-t_F)} + \left(n_{o1}^* + \dfrac{a_{21} n_{o2}^*}{a_{11}-a_{22}}\right) e^{-a_{11}(t-t_F)} \\ n_2^*(t) = n_{o2}^* e^{-a_{22}(t-t_F)} \end{cases} , \qquad (E.14)$$

having indicated by n_{o1}^* and n_{o2}^* the "final conditions" at time t_F.

Let us now suppose that the quantity to be analyzed is:

$$Q = n_2(t_F) = \begin{vmatrix} 0 & 1 \end{vmatrix} \begin{vmatrix} n_1(t_F) \\ n_2(t_F) \end{vmatrix} , \qquad (E.5)$$

so that the final condition results:

$$\begin{vmatrix} n_{o1}^* & n_{o2}^* \end{vmatrix} = \begin{vmatrix} 0 & 1 \end{vmatrix} . \qquad (E.6)$$

Besides, let us suppose that $n_{o2} = 0$. Setting $t_o = 0$, the solutions (E.2) and (E.4) become:

$$\begin{cases} n_1(t) = n_{o1} e^{a_{11} t} \\ n_2(t) = \dfrac{n_{o1} a_{21}}{a_{11}-a_{22}} \left(e^{a_{11} t} - e^{a_{22} t} \right) \end{cases} \qquad (E.7)$$

$$\begin{cases} n_1^*(t) = \dfrac{a_{21}}{a_{11}-a_{22}} \left[e^{-a_{11}(t-t_F)} - e^{-a_{22}(t-t_F)} \right] \\ n_2^*(t) = e^{-a_{22}(t-t_F)} \end{cases} \qquad (E.8)$$

By perturbing a_{11} by a quantity δa_{11} and adopting the time-dependent generalized perturbation method we obtain the expression:

$$\delta Q = \int_{t_o}^{t_F} dt \; \left| n_1^* \quad n_2^* \right| \left| \begin{matrix} \delta a_{11} & 0 \\ 0 & 0 \end{matrix} \right| \left| \begin{matrix} n_1 \\ n_2 \end{matrix} \right| =$$

$$= \int_{t_o}^{t_F} dt \; \left| n_1^* \quad n_2^* \right| \left| \begin{matrix} n_1 \delta a_{11} \\ 0 \end{matrix} \right| = \delta a_{11} \int n_1^* n \; dt =$$

$$= \delta a_{11} \, n_{o1} \left\{ \frac{a_{21} \, t_F}{a_{11}-a_{22}} \, e^{a_{11} t_F} - \frac{a_{21}}{(a_{11}-a_{22})^2} \left[e^{a_{11} t_F} - e^{a_{22} t_F} \right] \right\} . \qquad (E.9)$$

From the second one of the solutions (E.7) one could obtain, by differentiation with respect to a_{11} at time t_F, exactly the same expression above, which shows the validity (as well as the limits) of the proposed perturbation method.

Let us suppose now that at time t_1 the densities n_1 and n_2 are altered by fractions m_1 and m_2. In this case the real solution is given by the expressions (E.7) in the interval (t_o, t_1), while in the interval (t_1, t_F) it results:

$$\begin{cases} n_1(t) = (1 - m_1) n_{o1} \, e^{a_{11}(t)} \\ \\ n_2(t) = \dfrac{n_{o1} \, a_{21}}{a_{11}-a_{22}} \left\{ (1-m_1) \, e^{a_{11} t} + \left[(1-m_2)(e^{a_{11} t_1} - e^{a_{22} t_1}) - \right. \right. \\ \left. \left. - (1-m_1) \, e^{a_{11} t_1} \right] \; e^{a_{22}(t - t_1)} \right\} . \end{cases} \qquad (E.10)$$

Instead, the adjoint solution remains that given by the expressione (E.8)

in the interval (t_1, t_F), while in the interval (t_o, t_1) it results:

$$\begin{cases} n_1^*(t) = \dfrac{a_{21}}{a_{11}-a_{22}} \left\{ -(1-m_2)\, e^{-a_{22}(t-t_F)} + [\,] e^{-a_{11}(t-t_1)} \right\} \\ n_2^*(t) = (1-m_2)\, e^{-a_{22}(t-t_F)} \end{cases} \quad , \qquad \text{(E.11)}$$

having indicated by [] the expression:

$$\left[(1-m_1)(e^{-a_{11}(t_1-t_F)} - e^{-a_{22}(t_1-t_F)}) + (1-m_2)\, e^{-a_{22}(t_1-t_F)} \right] .$$

The perturbation δQ in this case becomes:

$$\delta Q = \delta a_{11} \int_o^{t_F} n_1^*\, n_1 dt = \delta a_{11} \int_o^{t_1} n_1^*\, n_1 dt + \delta a_{11} \int_{t_1}^{t_F} n_1^*\, n\, dt \quad =$$

$$= \delta a_{11}\, \frac{n_{o1}\, a_{21}}{a_{11}-a_{22}} \left\{ -\frac{1-m_2}{a_{11}-a_{22}}\, e^{a_{22}t_F} \left[e^{(a_{11}-a_{22})t_1} - 1 \right] + [\,]_{t_1} e^{a_{11}t_1} + \right.$$

$$\left. + (1-m_1)(t_F-t_1) e^{a_{11}t_F} - \frac{1-m_1}{a_{11}-a_{22}}\, e^{a_{22}t_F} \left[e^{(a_{11}-a_{22})t_F} - e^{(a_{11}-a_{22})t_1} \right] \right\} . \qquad \text{(E.12)}$$

We can easily verify also for this case that, by differentiating the second one of the expressions (E.10) at time t_F, we would obtain the some perturbative expression. From what shown above, the interest results proved of adopting the time-dependent generalized perturbation methods in complex cases, for which the analytical solution of the burn-up equations would be impractical, and rather a numerical approach results prefereable, when the sensitivity values are required of a given burn-up-

-dependent integral quantity with respect to the parameters appearing in the same equations.

REFERENCES

1. A. GANDINI, Nucl. Sci. Eng., 35, 141 (1969).